Higher du Bois and higher rational singularities for LCI varieties

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Overview

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3. Minimal Exponent for Local Complete Intersections (LCI)

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3. Minimal Exponent for Local Complete Intersections (LCI)
4. Brief description of mixed Hodge modules and local cohomology
5. Sketch of Proof and some Corollaries

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- For affine space $\mathbf{A}^r$, we will often denote a choice of coordinates on $\mathbf{A}^r$ by a subscript. So $\mathbf{A}^n_y$ has coordinates $y_1, \ldots, y_r$.
- We use $V(f_1, \ldots, f_r) \subseteq X$ to denote the subvariety defined by the regular functions $f_1, \ldots, f_r$ in X.

Hypersurface case: Log canonical threshold

- Let $H = V(f) \subseteq X$ be a hypersurface in the smooth variety X. Let $\pi : Y \to X$ be a strong resolution of singularities of the pair (X, H), i.e., a proper map with Y smooth which is an isomorphism over $X - H$ and so that $E = \pi^{-1}(H)$ has normal crossings support.

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- Numerical data: Let $K_{Y/X} = \sum_{i \in I} k_i E_i$ and $\pi^*(H) = \mathrm{div}(\pi^*(f)) = \sum_{i \in I} a_i E_i$, where E_i are prime divisors which are exceptional for π.

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- It is related to triviality of multiplier ideals $\mathcal{I}(f^\lambda)$, which are also defined via numerical data.

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- An interesting example is the cusp: $f = x_1^2 + x_2^3$. It satisfies $\operatorname{lct}(f) = \frac{5}{6}$.

Differential Operators

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- This is a *non-commutative ring* unless X is a point. Indeed, the commutator $[\partial_{x_i}, h] = \partial_{x_i}(h)$ need not be 0.

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- Consider the free, rank one $\mathcal{O}_X[s, \frac{1}{f}]$-module

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which we endow via the Leibniz and power rules an action of $\mathcal{D}_X$ (which commutes with s):

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Theorem (Bernstein,Kashiwara,Björk)

There exists a non-zero monic polynomial $b_f(s) \in \mathbf{C}[s]$ of least degree and an element $P(s) \in \mathcal{D}_X[s]$ such that

$$b_f(s)f^s = P(s)f^{s+1},$$

called the **Bernstein-Sato polynomial** *of f.*

Examples of Bernstein-Sato Polynomials

	Smooth	Normal Crossings	Cusp
f	x_1	$x_1 x_2$	$x_1^2 + x^3$
LCT:	1	1	$\frac{5}{6}$
$b_f(s)$:	$(s+1)$	$(s+1)^2$	$(s+1)(s+\frac{5}{6})(s+\frac{7}{6})$

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2. (Lichtin, Kollár) We see LCT as (negative of largest) roots of these polynomials.
3. (Kashiwara) All roots are negative and rational.
4. (Brainçon-Maisonobe) Only the smooth one has $b_f(s)$ actually equal to $(s+1)$.

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- Trivially we have $\mathrm{lct}(f) = \min\{1, \widetilde{\alpha}(f)\}$ and it is a positive rational number.
- Non-trivially: Saito showed $\widetilde{\alpha}(f) \leq \frac{n}{2}$ if f defines a singular hypersurface. If f defines a smooth hypersurface, we set $\widetilde{\alpha}(f) = +\infty$.

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- In a vague (Hodge theoretic) sense, this is a nice replacement for the de Rham complex $\Omega_Z^\bullet$.

Higher du Bois singularities

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- Jung, Kim, Saito and Yoon defined recently the class of *k-du Bois* singularities, which are those for which the map

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Theorem (JKSY, Mustață-Popa-Olano-Witaszek)

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$$\widetilde{\alpha}(f) \geq k+1 \iff H \text{ has } k\text{-du Bois singularities.}$$

Higher rational singularities

- A classical notion of singularity is *rational singularities*: let $\pi : \widetilde{Z} \to Z$ be a resolution of singularities. Then Z has *rational singularities* iff the natural map $\mathcal{O}_Z \to R\pi_*(\mathcal{O}_{\widetilde{Z}})$ is a quasi-isomorphism.

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- Kovács showed that rational singularities are du Bois. Saito showed that, for $H = V(f) \subseteq X$, H has rational singularities iff $\widetilde{\alpha}(H) > 1$.
- Recently, Friedman-Laza defined the notion of *k-rational singularities*. Using a resolution, one can construct a morphism

$$\underline{\Omega}_Z^k \xrightarrow{\psi_k} R\mathcal{H}om(\underline{\Omega}_Z^{\dim Z}, \omega_Z^\bullet).$$

  Then one requires Z be k-du Bois and for ψ_p to be a quasi-isomorphism for all $p \leq k$. For hypersurfaces, Saito shows equiv. to $\widetilde{\alpha}(f) > k + 1$.

Case of $Z = V(f_1, \ldots, f_r)$

- The notion of LCT immediately generalizes to Z defined by an ideal $(f_1, \ldots, f_r)$. In fact, one can define a Bernstein-Sato polynomial for $f_1, \ldots, f_r$: $b_{\underline{f}}(s)$, and the LCT is again the negative of the largest root of this polynomial.

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- Budur-Mustață-Saito related this polynomial to rational singularities of Z, if $\mathrm{codim}_X(Z) = r$. However, for the other classes of singularities, this is difficult (thus far, not possible) to do.
- To remedy this, we take inspiration from a result of Mustață:

Theorem (Mustață)

Let $g = \sum_{i=1}^r f_i y_i \in \mathcal{O}_Y$ where $Y = X \times \mathbf{A}_y^r$. Then

$$\widetilde{b}_g(s) = b_{\underline{f}}(s).$$

Definition of Minimal Exponent for Z

- Let $U = Y - (X \times \{0\})$. Assume $\operatorname{codim}_X(Z) = r$ (so Z is a *complete intersection*). We define the *minimal exponent* of Z as

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- Secondly, a simple computation shows that

$$\mathrm{Sing}(g) = (Z \times \{0\}) \cup \Sigma,$$

where Σ lies over Z_{sing}. Restricting to U removes the "trivial" part of this singular locus.

Properties and Example

Proposition

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Let $f_1, \ldots, f_r$ be weighted homogeneous polynomials on $\mathbf{A}_x^n$ of the same degree D. Let $w_1, \ldots, w_n$ be the weights of the variables $x_1, \ldots, x_n$, so that $\left(\sum_{j=1}^n w_j x_j \partial_{x_j}\right)(f_i) = Df_i$.

If $Z = V(f_1, \ldots, f_r)$ has codimension r and has only a singular point at 0, then $\widetilde{\alpha}(Z) = \frac{\sum_{i=1}^n w_i}{D}$. (This is already known for $r = 1$)

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Let $Z \subseteq X$ be a local complete intersection of pure codimension r. Then

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- To give a sketch of the proof, we need to vaguely describe what *mixed Hodge modules* on X are. These were defined by Saito.
- The category of mixed Hodge modules on X is an abelian category $\mathrm{MHM}(X)$ of finite length. It satisfies a "six functor formalism" in the sense of Grothendieck.

Hodge Modules

- For any smooth complex algebraic variety W, part of the data of a mixed Hodge module is a *bifiltered* $\mathcal{D}_W$-module:

$$(\mathcal{M}, F_\bullet\mathcal{M}, W_\bullet\mathcal{M}),$$

where $F_\bullet$ (the "Hodge filtration") is bounded below and consists of coherent $\mathcal{O}_W$-submodules and $W_\bullet$ (the "weight filtration") is finite and consists of $\mathcal{D}_W$-submodules.

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- Every morphism of mixed Hodge modules is a $\mathcal{D}_W$-linear map. It is automatically bi-strict with respect to F and W.
- If W is a point, then $\mathrm{MHM}(W)$ is equivalent to the category of (graded polarized) mixed Hodge structures.

V-filtrations

- Now let $W = X \times \mathbf{A}^r_t$. Kashiwara (following work of Malgrange) showed that every "regular holonomic" $\mathcal{D}_W$-module $\mathcal{M}$ admits a "V-filtration" along $t_1, \ldots, t_r$.

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- Initially, this filtration was indexed by $\mathbf{Z}$, but Saito refined it to a $\mathbf{Q}$-indexed filtration. In this way, it is discretely and left-continuously indexed (so there are countably many jumping numbers). Essentially, it attempts to diagonalize the Euler operator $\theta = \sum_{i=1}^r t_i \partial_{t_i}$.

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- The important properties are
  1. $t_i V^\lambda \mathcal{M} \subseteq V^{\lambda+1}\mathcal{M}$.
  2. $\partial_{t_i} V^\lambda \mathcal{M} \subseteq V^{\lambda-1}\mathcal{M}$.
  3. $\theta - \lambda + r$ acts nilpotently on $gr_V^\lambda \mathcal{M}$, where $V^{>\lambda}\mathcal{M} = \bigcup_{\beta>\lambda} V^\beta \mathcal{M}$.

Local Cohomology (mixed Hodge) module

- Returning to LCI $Z = V(f_1, \ldots, f_r) \subseteq X$, the middle-man in the proof is the local cohomology mixed Hodge module $\mathcal{H}^r_Z(\mathcal{O}_X)$. This is defined as the cokernel of the natural map

$$\bigoplus_{i=1}^{r} \mathcal{O}_X[\frac{1}{f_1 \ldots \hat{f}_i \ldots f_r}] \to \mathcal{O}_X[\frac{1}{f_1 \ldots f_r}].$$

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- Saito (for $r = 1$) and Mustață-Popa (in general) showed that

$$F_k \subseteq P_k.$$

Sketching the proof

- If $i : X \to X \times \mathbf{A}_t^r$ is the graph embedding along $f_1, \ldots, f_r$, we can consider the Hodge module $B_f = i_+\mathcal{O}_X$. It has easy to understand Hodge and weight filtrations. The interesting thing about it is its V-filtration along $t_1, \ldots, t_r$.

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- The "evaluate at -1" map $\mathbf{B} \to \mathcal{H}^r_Z(\mathcal{O}_X)$ sending $s_i \mapsto -1$ restricts to $V^r B_f \subseteq B_f$. It turns out that it descends to an isomorphism on the quotient

$$V^r B_f / \sum_{i=1}^r t_i V^{r-1} B_f. \tag{2}$$

Finishing the Sketch

- My work with Qianyu Chen shows that the quotient is even isomorphic to $\mathcal{H}^r_Z(\mathcal{O}_X)$ as a *mixed Hodge module*. In fact, the map described above, by general considerations, is one such isomorphism.

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- Mustață-Popa showed that Z has k-du Bois singularities iff $F_k = P_k$. We show that $\widetilde{\alpha}(Z) \geq r + k$ is equivalent to $F_k B_f \subseteq V^r B_f$, and under the map described above, this is equivalent to $F_k = P_k$.

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- We show that Z has k-rational singularities iff $F_k \cap W_{n+r} = P_k$ (which, of course, implies $F_k = P_k$, so Z has k-du Bois singularities). It is not hard to see that this latter condition is equivalent to $F_{k+1} B_f \subseteq V^{>(r-1)} B_f$. We show finally that this is equivalent to $\widetilde{\alpha}(Z) > r + k$, finishing the proof.

Some Corollaries

- For LCI Z, k-du Bois implies $(k-1)$-rational.
- (MP) If LCI Z has k-du Bois singularities, then $\operatorname{codim}_Z(Z_{sing}) \geq 2k+1$.
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Sketch of Proof.

By the restriction result, we can slice by general hyperplanes to assume Z has isolated singularities. Then we must show that $\dim Z = d \geq 2k+2$. In analogy with Saito's upper bound, we know for $x \in Z_{sing}$

$$\widetilde{\alpha}_x(Z) \leq \dim X - \frac{1}{2}\dim_{\mathbf{C}} T_x Z,$$

and by $x \in Z_{sing}$, we have $\dim_{\mathbf{C}} T_x Z \geq d+1$. Then use $\widetilde{\alpha}_x(Z) > r+k$ to conclude $d > 2k+1$. □